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Middle School Mathematics Teachers' Knowledge of Students' Understanding of Core Algebraic Concepts: Equal Sign and Variable

Pamela Asquith, Ana C. Stephens, Eric J. Knuth, and Martha W. Alibali
University of Wisconsin-Madison

This article reports results from a study focused on teachers' knowledge of students' understanding of core algebraic concepts. In particular, the study examined middle school mathematics teachers' knowledge of students' understanding of the equal sign and variable, and students' success applying their understanding of these concepts. Interview data were collected from 20 middle school teachers regarding their predictions of student responses to written assessment items focusing on the equal sign and variable. Teachers' predictions of students' understanding of variable aligned to a large extent with students' actual responses to corresponding items. In contrast, teachers' predictions of students' understanding of the equal sign did not correspond with actual student responses. Further, teachers rarely identified misconceptions about either variable or the equal sign as an obstacle to solving problems that required application of these concepts. Implications for teacher professional development are discussed.

INTRODUCTION

Much research on students' understanding of algebra has documented difficulties and misconceptions. Results from the sixth mathematics assessment of the National Assessment of Educational Progress (Kenney & Silver, 1997), for example, indicate that twelfth-grade students have difficulty solving all but the simplest

Correspondence should be sent to Pamela Asquith, Wisconsin Center for Education Research, University of Wisconsin-Madison, 1025 W. Johnson Street, Madison, WI 53706.
E-mail: pasquith@wisc.edu

algebraic equations and inequalities and have great difficulty translating from verbal to symbolic representations. Kieran (1992) reports that most students do not acquire any sense of the structural aspects of algebra and, “in order to cover their lack of understanding…resort to memorizing rules and procedures and…eventually come to believe that this activity represents the essence of algebra” (p. 390).

In response to students’ inadequate understandings of and preparation in algebra, as well as in recognition of algebra’s role as a gatekeeper to future educational and employment opportunities (Ladson-Billings, 1998; Moses & Cobb, 2001; National Research Council [NRC], 1998), many in the mathematics education community have called for algebra reform (e.g., Kaput, 1998; Olive, Izsak, & Blanton, 2002). Algebra reform, however, involves more than simply “fixing” the traditional ninth-grade course. There is an emerging consensus that algebra reform requires reconceptualizing the nature of algebra in school mathematics and treating the subject as a continuous K–12 strand. “By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school” (National Council of Teachers of Mathematics [NCTM], 2000, p. 37). The impact of this reconceptualization has been most apparent in elementary school, where mathematics educators have recently made concerted efforts to integrate algebraic ideas (e.g., Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput, Carraher, & Blanton, 2007).

The integration of algebra in the elementary grades implies recognition of algebraic reasoning as more than mastery of equation manipulation (Carpenter & Levi, 2000; Schifter, 1999) and involves viewing algebra as including new forms of reasoning accessible to students across the grades. Introducing algebraic ideas to students earlier, however, presents many challenges, including learning more about the development of students’ early algebraic reasoning, designing supportive curricula, and developing teacher knowledge and practice that will enable teachers to foster connections between arithmetic and algebraic forms of reasoning. These challenges are particularly relevant at the middle school level, at which time the transition from arithmetic to algebraic thinking is arguably most salient.

Of these challenges, teachers’ knowledge has been identified as an important determinant of their classroom practices (Borko & Putnam, 1996) and, ultimately, has major implications for what students learn (Hill, Rowan, & Ball, 2005). Yet, to date, there has been little research focused on middle school teachers’ knowledge of algebra as it pertains to integrating algebraic ideas into the middle school curriculum or to the development of students’ algebraic reasoning. Accordingly, the focus of the research reported in this article is middle school teachers’ knowledge of students’ algebraic reasoning as revealed by their predictions of student thinking about two core algebraic concepts—the equal sign and variable.

TEACHER KNOWLEDGE OF STUDENTS' MATHEMATICAL THINKING

Early research on teachers' mathematical knowledge involved counting the number of mathematics courses completed or degrees earned. Such research failed to establish a clear relationship between knowledge and student achievement (Grossman, Wilson, & Shulman, 1989). Since that time, researchers have realized the importance of looking more closely at the *substance* of teachers' knowledge, particularly knowledge assumed necessary for teaching (Ball, Lubienski, & Mewborn, 2001; Ma, 1999). The construct of pedagogical content knowledge, "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman, 1986, p. 9) brought attention to the importance of teachers' knowledge of students' understandings, conceptions, and misconceptions of particular topics in a subject matter. And indeed, such attention to teachers' knowledge of student thinking is reflected in the work of a number of scholars (e.g., Ball & Cohen, 1999; Kazemi & Franke, 2004).

The seminal work of Carpenter and colleagues (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998) with Cognitively Guided Instruction [CGI] established a strong connection between teachers' knowledge of student thinking and students' achievement in the domain of whole-number arithmetic. Teachers taking part in professional development focused on research-based knowledge of student thinking had students who exceeded peers in control classes in number fact knowledge, problem solving, reported understanding, and reported confidence in their problem-solving abilities. Similarly, Kazemi and Franke (2004) introduced CGI principles and terminology to teachers with whom they worked and also found positive changes in the teachers' thinking and instructional practices.

A limitation of much of the research regarding teachers' knowledge of student thinking is that the research has focused primarily on the domains of whole number and rational number in the early elementary grades. Given the recent calls for integrating algebraic reasoning throughout the K–8 curriculum, however, researchers have begun to move toward teachers' knowledge of student thinking in the domain of early algebra in elementary school mathematics (e.g., Blanton & Kaput, 2003; Carpenter et al., 2003; Kaput et al., 2007; Stephens, 2006). Yet, notwithstanding the importance of this more recent research, fewer efforts have focused on teachers' knowledge of student thinking about algebraic ideas in the middle grades—a period that marks a significant transition from the concrete, arithmetic reasoning of elementary school mathematics to the increasingly complex, abstract algebraic reasoning required for high school mathematics and beyond.

Nathan and Koedinger (2000) have given this topic some attention, investigating teachers' beliefs about the relative problem-solving difficulty students would encounter on a set of tasks that varied in presentation format (words vs. equations) and placement of the unknown (arithmetic vs. algebraic). They concluded that teachers hold a "symbol-precedence" view of students' development in this domain, believing that symbolic reasoning precedes verbal reasoning. They argue such a view is likely due to the influence of traditional textbook organization, in which "word problems" are presented after comparable symbolic problems. High school teachers were particularly apt to hold such a view, perhaps due to an "expert blindspot" resulting from more extensive content training and less appreciation for the struggles students might experience learning algebra (see Nathan & Petrosino, 2003). Nathan and Koedinger (2000) suggest that tasks such as the ones employed in these studies be used in professional development and teacher education settings to confront teachers' misconceptions about students' mathematical learning and development.

Our aim in this article is to build on Nathan and Koedinger's (2000) work on teacher knowledge of students' algebraic thinking by focusing specifically on two core concepts—the equal sign and variable—with which students have traditionally struggled, and which are critical to algebra understanding. Results from the Supporting the Transition from Arithmetic to Algebraic Reasoning [STAAR] project indicate that middle school students' understanding of these core concepts correlates with their success solving algebraic problems, the strategies they use to solve the problems, and the justifications they provide for their solutions (Alibali, Knuth, Hattikudur, McNeil, & Stephens, this issue; Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006).

As we learn more about middle school students' algebraic thinking, a logical next step is to consider what conceptions middle school teachers have about their students' thinking in this domain. To this end, our study compared teachers' predictions of student thinking on tasks with actual student performance on those same problems. Our specific research questions were the following:

1. What are middle school teachers' conceptions of student thinking around issues of the equal sign and variable?
2. How do these conceptions compare to actual student performance on equal sign and variable tasks?

Before sharing the results of this study, we will first briefly present an overview of what is known about student understanding of these two concepts.

STUDENT UNDERSTANDING OF THE EQUAL SIGN AND VARIABLE

Numerous studies investigating elementary and middle school students' conceptions of the equal sign have converged on similar conclusions: many students lack a relational understanding of the equal sign (i.e., the understanding that the equal sign represents an equivalence relation between two quantities) and instead view it as signifying the answer or result of an arithmetic operation (e.g., Behr, Erlwanger, & Nichols, 1980; Kieran, 1981; Falkner, Levi, & Carpenter, 1999; Rittle-Johnson & Alibali, 1999). A relational view of the equal sign is important when working with algebraic equations and is necessary for understanding that the transformations performed in solving an equation preserve the equivalence relation (Alibali et al., this issue; Kieran, 1992; Knuth et al., 2005; Knuth et al., 2006).

Research on student thinking about variable has likewise shown that many students' conceptions are inadequate, particularly with respect to the use of literal symbols in algebra (e.g., Küchemann, 1978; Usiskin, 1988). Student misunderstandings include viewing variables as abbreviations or labels rather than as letters that stand for quantities, assigning values to letters based on their positions in the alphabet, and otherwise being unable to operate with algebraic letters as varying quantities rather than specific values (Küchemann, 1978).

METHOD

Participants

An invitation to participate in our study was sent to all middle school mathematics teachers in a small urban district in the American Midwest (45% minority, 41% low-income student population), where our student research was conducted. Only 15 teachers who had participated in a professional development course connected to the STAAR project were not invited. Of the 85 teachers invited, 20 from nine middle schools agreed to participate. Each was paid $20 for his or her participation. Prior to the interview, participants were asked to respond to an optional written survey addressing their educational backgrounds, prior teaching experience, curricula use, and professional development activities.

The 20 teachers included 10 sixth-grade, 6 seventh-grade, and 4 eighth-grade teachers. Four teachers taught only mathematics, while the other 16 also taught science, social studies, or language arts. Teaching experience ranged from 5 to 31 years with 14 participants having taught 11 years or more. Degrees held were primarily bachelor's degrees in elementary education. One participant held a master's degree in math education. Sixteen participants responded to a question about how recently they had enrolled in a mathematics course. Of these, 7 reported that it had been at least 10 years since their last math course and 9 reported that it had been 20 years or more. At the time of this study, 19 teachers were using the *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Philips, 1998) curriculum and one teacher was using the *Mathematics in Context* (Romberg et al., 1998) curriculum. All teachers had participated in district-led curriculum workshops intended to acquaint teachers with the *CMP* curriculum. Four teachers had taken professional development classes taught by the University of Wisconsin-Madison's mathematics department. These courses focused on mathematical content but did not address student understanding of algebraic concepts.

Data Collection

After completing the written survey, each teacher participated in an hour-long videotaped interview framed by a portion of the STAAR project's longitudinal written assessment of middle school students' understandings of various algebraic concepts. The interviews were conducted between December and February of the 2004–2005 academic year and took place at teachers' schools. The student data to be compared with teacher predictions were collected from sixth- through eighth-grade students attending a middle school (39% minority, 41% low-income student population) in the same district from which the participants were drawn. Four tasks—two addressing the concept of equal sign and two addressing the concept of variable—were selected from this written assessment to form the focus of the teacher interview (see Figure 1). The written assessment consisted of three forms with some overlap of items; all 373 students received the *equal sign definition* and *literal symbol interpretation* tasks, 128 students received the *equivalent equations* task, and 122 students received the *which is larger* task.

Teachers were presented each of these four tasks and were asked the following questions:

1. What answers—correct or incorrect—would you expect your sixth- (or seventh- or eighth-) grade students to give to this problem and what strategies might they have used to get those answers?
2. What do you believe a student who gives that answer might be thinking?
3. Suppose you gave this problem to 100 sixth- (or seventh- or eighth-) grade students from across the school district, including a wide range of ability levels. Could you indicate how many you expect would use each strategy?
4. Could you explain your reasoning behind the assignment of these numbers?

1. Equal sign definition task

   $3 + 4 = 7$

   ↑

   The arrow above points to a symbol.
   What is the name of the symbol?
   What does the symbol mean?

2. Equivalent equations task

   Is the number that goes in the □ the same number in the following two equations?

   $2 \times \square + 15 = 31$ $\qquad$ $2 \times \square + 15 - 9 = 31 - 9$

3. Literal symbol interpretation task

   The following question is about this expression:

   $2n + 3$

   ↑

   The arrow above points to a symbol.
   What does the symbol stand for?

4. Which is larger task

   Can you tell which is larger, $3n$ or $n + 6$?
   Please explain your answer.

FIGURE 1 Equal sign and variable tasks.

The *equivalent equations* and *which is larger* tasks were presented prior to the *equal sign definition* and *literal symbol interpretation* tasks to avoid confounding teacher responses (e.g., teachers may have inferred they should make a connection between defining the symbol and solving a problem with that same symbol if asked to define the symbol first). The interviewer recorded all answers—correct and incorrect—that teachers believed students at their respective grade levels would give and then asked teachers to identify the strategies and student thinking behind each of these responses. As teachers were asked to assign percentages to each response (question 3 above), they were able to review the interviewer's notes to verify that an exhaustive and representative list of their responses had been compiled. Finally, the interviewer asked teachers to explain their reasoning for the distribution of student responses and strategies provided.

Data Analysis

Teacher responses to question 1 for each of the four tasks were coded with the same scheme designed to describe student responses to the corresponding written assessment items (see Figure 2). Two STAAR project staff members who had prior experience coding the student data for these items completed the coding. To assess reliability, a second coder scored 50% of the data. Agreement between coders for predictions of student strategies was 99% on the *equal sign definition* task, 85% on the *equivalent equations* task, 95% on the *literal symbol interpretation* task, and 100% on the *which is larger* task.

For each task, the proportion of student responses of each type predicted by teachers was averaged across all participants at each grade level, including zeroes for teachers who did not suggest a particular response. The total of teacher estimates in response to question 3 sometimes exceeded 100% because students might employ more than one strategy. In such cases, teacher predictions for each student response were scaled to total 100% so they could be compared to the student responses. For the *equivalent equations* and *which is larger* tasks—which potentially required students to apply their knowledge of the equal sign and variable—interview transcripts were analyzed for teacher identification of mathematical knowledge necessary for success and potential misconceptions that would hinder student performance. Sometimes teachers offered this information spontaneously, but they were also prompted to do so by the interviewer when asked to describe the student thinking underlying each strategy and to provide a rationale for their predictions (questions 2 and 4 above). All instances of both requisite knowledge and misconceptions proposed for each task were identified. A second coder followed the same procedure for 20% of the data. Agreement between coders was 88% for the *equivalent equations task* and 86% for the *which is larger* task.

RESULTS

We first compare teacher predictions of the proportion of students who would offer a particular response with student performance on the two variable and two equal sign tasks. We then report the knowledge teachers identified as necessary for students to succeed on the *equivalent equations* and *which is larger* tasks, as well as the misconceptions they anticipated would hinder students' performance on these tasks.

Equal sign definition task:

- *Operational*: Response expresses the idea that the equal sign means "add the numbers" or "the answer."
- *Relational:* Response expresses the idea that the equal sign means "the same as."

Equivalent equations task:

- *Solve and compare:* The two equations are solved using any method and the solutions compared, or one equation is solved and substitution is used to see if the solution to the second equation is identical.
- *Recognize equivalence*: Equivalence of the two equations is recognized without solving the equations.
- *Answer after the equal sign*: Response expresses the idea that the solution to the problem is the value of the right side of the equal sign or the first number to the right of the equal sign.

Literal symbol interpretation task:

- *Multiple values:* Response expresses the idea that the symbol can stand for any number.
- *Specific number:* Response expresses the idea that the symbol can stand for one specific number only.
- *Object*: Response expresses the idea that the symbol is an object or word beginning with the letter *n*.
- *Unknown digit*: Response expresses the idea that the symbol stands for an unknown number that is a single digit (either 1–9 or 0–9).
- *Concatenation*: Response expresses the idea that the symbol stands for the digit in the ones place (so 2*n* is "twenty something").

Which is larger task:

- *Variable:* Response expresses the idea that one cannot determine which quantity is larger because the variable can take on multiple values.
- *Single-value:* A single value is tested and a conclusion is drawn on that basis; thus, a student's conclusion might vary depending on the value tested.
- *Operation:* Response expresses the idea that one type of operation leads to larger values than the other (e.g., "Three times, because you get a larger amount with times").

Global codes used across tasks:

- *Don't know, No response, Other*: Responses are identified but do not fit any of the above categories.

FIGURE 2 Coding scheme for equal sign and variable tasks.

Comparing Teacher Predictions with Student Performance on the Variable Tasks

Literal Symbol Interpretation Task. Teachers' predictions of student performance on the *literal symbol interpretation* task (see Figure 1), in which students are asked what the symbol n in $2n + 3$ stands for, aligned relatively well with actual student performance across all grades (see Table 1). Most impressive were teacher predictions regarding the proportion of students at each grade level who would understand that n could stand for multiple values and the proportion of students who would lack the requisite knowledge to respond to this question. Eighteen of the 20 teachers were in agreement that at least some percentage of their students would give a multiple-values interpretation of n (e.g., n can stand for any number) and 13 of the 16 sixth- and seventh-grade teachers were in agreement that some students would be stumped by this problem, while all 4 eighth-grade teachers agreed that students would be able to provide a response (see Table 2). Teachers also accurately predicted the proportion of students who would hold misconceptions about the symbol n (represented by *specific number*, *object,* and *other* in Table 1), although there was some variation in the types of misconceptions evident in student performance and teacher predictions. Seven teachers believed that students might think n could only stand for one specific number and six teachers also recognized that some students might think n stands for a word or an object (see Table 2). For example, Kevin,* an eighth-grade teacher, stated, "These [students] have no idea letters and variables are connected so they might think $2n$ stands for two nickels." Two sixth-grade teachers recognized that lacking an understanding of concatenation could result in the misinterpretation of $2n$ as standing for "twenty something."

TABLE 1
Proportion of Student Responses of Each Type and Average Teacher Prediction of Proportion of Student Responses of Each Type in Response to the *Literal Symbol Interpretation* Task

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Interpretation						
Multiple value	0.46	0.44	0.63	0.62	0.76	0.81
Specific number	0.02	0.12	0.03	0.14	0.01	0.16
Object	0.04	0.10	0.11	0.02	0.09	0.01
Other	0.19	0.06	0.09	0.10	0.09	0.02
No response/ Don't know	0.29	0.28	0.14	0.12	0.05	0.00

*All names are pseudonyms.

TABLE 2
Number of Teachers Who Predicted Each Student Response by Task

Task	*Response*	*Grade level* 6th (n = 10)	7th (n = 6)	8th (n = 4)	Total (n = 20)
Literal symbol interpretation	Multiple value	9	5	4	18
	Specific number	4	1	2	7
	Object	3	2	1	6
	Other	3	3	1	7
	No response/Don't know	9	4	0	13
Which is larger-judgment	Can't tell	9	3	4	16
	3n	5	5	2	12
	n + 6	6	2	1	9
	No response/Don't know	8	6	2	16
Which is larger-justification	Variable	9	3	4	16
	Operation	6	4	2	12
	Single value	3	1	1	5
	Other	4	2	0	6
	No response/Don't know	8	5	3	16
Equal sign definition	Relational	10	6	4	20
	Operational	5	2	1	8
	Other	3	1	2	6
	No response/Don't know	1	1	0	2
Equivalent equations	Recognize equivalence	3	0	4	7
	Solve and compare	9	6	4	19
	Answer	0	1	0	1
	Other	4	2	0	6
	No response/Don't know	7	3	1	11

Which is Larger Task. There were more discrepancies between student performance and teacher predictions on the *which is larger* task (see Figure 1) than on the *literal symbol interpretation* task. Sixth- and eighth-grade teachers' predictions matched more closely than seventh-grade teachers' predictions with regard to the proportion of students who gave the correct judgment *you can't tell which is larger* (see Table 3). Seventh-grade teachers tended to underestimate the proportion of students who would respond in that way. However, considering the proportion of *can't tell* responses that were correctly justified by a *variable* explanation (see Table 4), seventh-grade teachers' predictions more accurately mirrored student understanding. For example, Sandra, a seventh-grade teacher, stated, "There will be some kids who do know enough in this case who would say you cannot solve this question because we do not know what *n* is." The sixth- and eighth-grade teachers, on the other hand, underestimated the proportion of students who would hold misconceptions, expecting *variable* justifications to accompany correct judgments.

A higher proportion of students incorrectly judged that $n + 6$ is larger than teachers predicted, especially at sixth grade where teachers expected more students to provide no response (see Table 3). Sixth- and eighth-grade teachers were relatively accurate in their predictions of the proportion of students who would think that $3n$ is larger. However, at seventh grade, teachers predicted that 37% of seventh-graders would think $3n$ is larger, while only 5% of students actually did. Sixth- and seventh-grade teachers predicted that the incorrect judgments, $3n$ and $n + 6$, would be justified by thinking that multiplication always results in larger numbers or that addition of 6 is greater than 3:

> [Students] will look at it and think, multiplying I'm going to get a bigger [number] than when I am adding. (*Beth, grade 7 teacher*)

> Yes, I think [students] would think that $n + 6$ is bigger because they can see you're adding 6 onto whatever number it is. With $3n$ I don't think they would be sure if the number was supposed to be getting bigger or not. (*Sarah, grade 6 teacher*)

However, although overall, 38% of the sixth- and seventh-graders gave the $n + 6$ or $3n$ response, only 4% of their justifications were based on the operation (see Table 5). Sixty-seven percent of the $3n$ responses were not justified or lacked a clear explanation and 44% of the $n + 6$ responses were justified by the explanation that 6 is greater than 3 (included in the *other* category in Table 5), indicating that students

TABLE 3
Proportion of Student Responses of Each Type and Average Teacher Prediction of Proportion of Student Responses of Each Type for the Judgment of *Which is Larger*

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Judgment						
Can't tell	0.18	0.28	0.54	0.20	0.64	0.76
$3n$	0.13	0.06	0.05	0.37	0.16	0.06
$n + 6$	0.41	0.18	0.18	0.03	0.16	0.00
Other	0.00	0.06	0.00	0.12	0.00	0.00
No response/ Don't know	0.28	0.42	0.23	0.28	0.04	0.18

TABLE 4
Proportion of Student Responses and Average Teacher Prediction of Proportion of Correct Judgments Accurately Justified by a Variable Explanation on the *Which is Larger* Task

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Justification						
Variable	0.08	0.28	0.39	0.34	0.53	0.75

TABLE 5
Proportion of Student Responses of Each Type and Average Teacher Prediction of Proportion of Student Responses of Each Type for the Justification of *Which is Larger*

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Justification						
Variable	0.10	0.28	0.52	0.18	0.60	0.74
Operation	0.00	0.19	0.05	0.36	0.09	0.06
Single value	0.03	0.07	0.05	0.12	0.04	0.02
Other	0.42	0.04	0.15	0.11	0.16	0.00
No response/ Don't know	0.45	0.42	0.23	0.23	0.11	0.18

who had difficulty tended to focus on the numbers present in the task rather than on the operations.

Students who provided examples to justify their judgment of which is larger substituted a single value for n or chose multiple values and then compared the two quantities. This strategy was mentioned by all 20 teachers. An additional strategy mentioned by five teachers across the grade levels was that students would figure out the value of n when $3n$ and $n + 6$ were equivalent and explain that $3n$ would be less than $n + 6$ when n was less than 3 and larger when n was greater than 3. However, only one student response across all grades exhibited this more sophisticated strategy.

In summary, teachers' predictions of students' understanding of variable aligned well with student performance on the *literal symbol interpretation* task, with the majority ($n = 18$) of teachers accurately predicting the proportion of students who would interpret a literal symbol to stand for multiple values. Predictions on the *which is larger* task, on the other hand, in which students were required to apply this knowledge, proved less accurate.

We turn next to teachers' predictions in response to the two equal sign tasks, which differed substantially from actual student performance. We discuss each task in turn.

Comparing Teacher Predictions with Student Performance on the Equal Sign Tasks

Equal Sign Definition Task. Students tend to progress with grade level from an operational to a relational view of the equal sign (Alibali et al., this issue). This progression, however, does not occur as quickly as teachers predicted (see Table 6). Teachers predicted that students at all grade levels would have a stronger relational understanding of the equal sign than was actually demonstrated by student responses. Instead, an operational view was more prevalent in the student responses than teachers predicted. A few teachers assumed that almost all students would hold a relational view:

> I think 100% would get the correct answer in sixth grade. And I guess my reasoning is that I have not ever had a kid at sixth grade who did not know what this [the equal sign] meant. (*Anne, grade 6 teacher*)

> Almost 99% [of eighth-grade students] across the district would say [the equal sign means] the left side equals the right side [of the equation] because they've used the equal sign a lot since kindergarten and have had experience working with equations. (*Kevin, grade 8 teacher*)

TABLE 6
Proportion of Student Responses of Each Type and Average Teacher Prediction of Proportion of Student Responses of Each Type in Response to the *Equal Sign Definition* Task

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Definition						
Relational	0.29	0.53	0.37	0.73	0.44	0.62
Operational	0.53	0.34	0.48	0.20	0.43	0.15
Other	0.14	0.10	0.12	0.04	0.11	0.23
No response/ Don't know	0.04	0.03	0.03	0.03	0.02	0.00

The contrast was most striking at seventh grade, where the average of the six teachers' predictions were that 73% of students would give a relational definition of the equal sign, but only 37% actually did (see Tables 2 and 6). Although teachers overestimated the proportion of students who would provide a relational definition of the equal sign, their descriptions of the student thinking underlying the possible definitions indicated that they understood the distinction between operational and relational views:

> I think more kids would say, "Well, you do 3 + 4 and that gives you 7," as though it is an operation. That the addition is causing the 7. They will see it as a causative, not a balance…. Others would be able to say it means what's on the left side is equal to what's on the right. That is what [the equal sign] really means. (*Linda, grade 6 teacher*)

Equivalent Equations Task. Student performance and teacher predictions in response to the *equivalent equations* task are shown in Table 7. Three of the 10 sixth-grade teachers accurately predicted that a small percentage of students would recognize the equivalence of the two equations without solving them first. For example, Angela, a sixth-grade teacher, stated, "I would guess that 20% might be able to see the equality by looking at it by seeing that the only thing that changed on either side of the equal sign was the subtract 9."

Although 20% of the seventh graders recognized the equivalence, seventh-grade teachers predicted that none of them would and eighth-grade teachers also slightly underestimated the proportion of students who would (see Table 7). Nineteen teachers across the grade levels predicted a greater percentage of students would use a *solve and compare* strategy than actually did. For example, "We would

TABLE 7
Proportion of Student Responses of Each Type and Average Teacher Prediction of Proportion of Student Responses of Each Type in Response to the *Equivalent Equations* Task

	Grade level					
	6th		*7th*		*8th*	
	Student	*Teacher*	*Student*	*Teacher*	*Student*	*Teacher*
Strategy						
Recognize equivalence	0.14	0.09	0.20	0.00	0.31	0.23
Solve and compare	0.40	0.61	0.30	0.75	0.31	0.68
Answer after the equal sign	0.07	0.00	0.13	0.06	0.16	0.00
Other	0.32	0.15	0.27	0.08	0.20	0.00
No response/ Don't know	0.07	0.15	0.10	0.11	0.02	0.09

have kids who would look at this and they would work out both problems and see if in fact they got the same answer" (*Sophie, grade 8 teacher*).

These results seem to contradict teachers' predictions that more than half of students at their respective grade levels would hold a relational view of the equal sign (see Table 6). In fact, as will be discussed further, teachers did not consider whether an operational or relational view of the equal sign might shape student thinking about this task. Other noteworthy observations include some teachers' failure to provide a correct response to this item ($n = 5$) or identify the *recognize equivalence* strategy ($n = 13$) as a potential solution (see Table 2).

We turn next to the knowledge teachers identified as necessary for students to succeed with the *which is larger* and *equivalent equations* tasks as well as the misconceptions they anticipated would hinder students' performance on these tasks.

Teachers' Identification of Mathematical Knowledge Relevant to Tasks

Throughout teachers' discussions of predicted responses to the *which is larger* and *equivalent equations* tasks, they identified mathematical knowledge and misconceptions concerning variable, equality, and other concepts they believed would be relevant. All mathematical concepts identified by 25% or more of the participants ($n = 5$) will be discussed along with representative interview excerpts.

Which is Larger Task. When teachers described the types of judgments and justifications students would provide in response to the *which is larger* task, they identified mathematical understandings required to compare the expressions $3n$ and $n + 6$ as well as misconceptions they believed might underlie students' errors. The most often cited requirement, noted by 18 of the 20 teachers, was that students would need to understand the notation for implicit multiplication. Others included the understanding that one should use the same value for n in each expression and test more than one value ($n = 15$), knowledge of how to substitute numbers in place of n ($n = 10$), knowledge that n stands for a number ($n = 6$), and the understanding that letters can represent numerical values ($n = 5$).

It is noteworthy that teachers more often cited knowledge of notation for implicit multiplication and procedural knowledge about evaluating expressions as key to student success than students' deeper understandings about variables. In addition, only two of the six teachers who identified "knowing that n stands for a number" as an important concept for this task specified that students would require a multiple-values interpretation of variable. The most prevalent misconception mentioned by teachers across the grades was that multiplication always produces larger results than addition ($n = 9$).

> You know, I think most kids come out of elementary school not really thinking in terms of multiplication and what it really does…. They're more familiar with addition that you add on so then you get bigger and they don't really understand…what multiplication's doing. (*Alice, grade 7 teacher*)

> I think the overriding thing would be, multiplication is bigger—it's always bigger. I heard my elementary teacher say it's always bigger so…I think it would deceive them. (*Angela, grade 6 teacher*)

Five teachers believed that students would base their judgment on only one value of n. This error was not attributed to the misconception that n can only stand for one particular number but rather to some students thinking that math problems have only one answer.

> I would have very few kids who would give me more than one example. If they just happen to choose 3 and look if they are the same, they'll stop.... I will have some that will go, "Let's try 4.… Now I have a difference in my answer.…" There will be a few of those kids but on the whole, once they find one answer, and it supports what they think, they'll stop. (*Karen, grade 7 teacher*)

Only three teachers mentioned students' lack of understanding about letters as variables as an obstacle to solving the problem. This is in stark contrast to teachers' predictions of students' interpretations of n in $2n + 3$ (the *literal symbol interpretation* task), in which 12 teachers identified a variety of misconceptions.

Equivalent Equations Task. In predicting student responses to the *equivalent equations* task (see Figure 1) and justifying these predictions, teachers identified several mathematical concepts they believed students would need to grasp to be successful with particular strategies. The most common statement—made by 17 participants—was that students would need to be able to substitute a numerical value in place of the box and then evaluate the resulting expression. This was not surprising given the large number of participants ($n = 19$) who identified *solve and compare* as a solution strategy students would use. More specifically, these participants believed students would use a "guess-and-test" method to solve one or both equations.

Fourteen participants noted that students would need to be able to use some sort of symbol manipulation or traditional equation solving technique:

> This is what I expect from students: Nine subtract from 31, and you would get 22. Then, 15 subtract 9 and you get 6. Then 2 times x…plus 6 equals 22. Then…they would subtract 6 from both sides. They have to do it to both sides, that's the golden rule of working with equations. And, 22 reduce 6 they get 16 and $2x$ equals 16. They divide both sides by 2 to get the answer. (*Kevin, grade 8 teacher*)

The belief that such equation-solving skills would need to be utilized by students again aligns with participants' frequent mention of *solve and compare* as a strategy students would use.

Other mathematical concepts identified as relevant to success on the *equivalent equations* task included an understanding of the "box" notation ($n = 9$) and knowledge of order of operations ($n = 8$), despite the fact that multiplication appears before addition in this task.

Six participants recognized that success on this task might be related to an understanding of the equal sign as a relational symbol:

> I've had a number of kids who have said that they see the equal sign as being a fulcrum and a balance, a scale, a pan scale. (*Jill, grade 7 teacher*)

> We always talk about equations being like a balance. So I need to get this side to balance with that 22. (*Susan, grade 8 teacher*)

A somewhat related statement, made by an additional four participants, was that students would need to recognize that the "minus 9s" on each side of the equation "cancel out" or "don't change anything."

Consistent with the *which is larger* task, participants rarely identified misconceptions students might hold that would hinder success on the *equivalent equations* task. Specifically, only two participants—both sixth-grade teachers—mentioned that students might hold an operational view of the equal sign:

> [Students will think] that they are two separate problems that need to be worked out. Most of my kids believe that the equals sign does something. I don't think they would see [the two equations] as a balancing thing. (*Linda, grade 6 teacher*)

> I don't think sixth-graders have a good sense of equality and that taking away 9 is going to balance each side of this equation and therefore the number will be the same.... Some of my kids would not understand why it's 31-9 on the right-hand side of the equation, because they are very used to seeing one number there. (*Angela, grade 6 teacher*)

The findings regarding teachers' predictions of student responses to the variable and equal sign tasks and the mathematics concepts they identified as relevant to the applied tasks are simultaneously encouraging and discouraging. The overall results will be discussed below, and implications for teacher professional development will be considered.

DISCUSSION

The teachers in this study experienced varying degrees of success when asked to predict how middle school students would respond to tasks addressing their understanding of the concepts of variable and the equal sign and applications of these concepts. With regard to variable, they understood that some middle school students view these symbols as objects or believe that they must stand for only one specific value. When asked to consider the concept of equality, most teachers knew that some students think of the equal sign as meaning "give the answer." However, the extent of this misconception was not accurately anticipated, with teachers predicting many more students would give a relational definition of the equal sign than was actually the case.

An understanding of the connections between the given tasks in each category was also lacking. First, concerning variable, although teachers predicted a large proportion of students would hold a multiple-values interpretation, they underestimated the proportion of students who would use this understanding to evaluate the algebraic expression presented in the *which is larger* task. In fact, few identified it as a concept key to solving this task, citing instead notation for implicit multiplication and students' misconceptions about multiplication as stumbling blocks. Second, concerning the equal sign, although teachers predicted a large proportion of students would hold a relational understanding of this symbol, they underestimated the proportion of students who would use this understanding to recognize equation equivalence. Furthermore, they did not consider how holding an operational view of the equal sign might hinder performance.

Prior analyses of student performance suggest a relationship between symbol understanding and success on related algebraic tasks. Knuth et al. (2005) found that a multiple-values response to the *literal symbol interpretation* task was associated with success on the *which is larger* task, and that a relational view of the equal sign was associated with success on the *equivalent equations* task. Alibali et al. (this issue) additionally found that the likelihood that a student would use the *recognize equivalence* strategy in eighth grade was greater had he or she acquired a relational understanding of the equal sign in sixth or seventh grade, suggesting it matters *when* students acquire a relational understanding of the equal sign. That teachers failed to see these connections is not necessarily surprising, given these tasks are not ones typically posed to students. This result nevertheless implies a need for focus in mathematics teacher professional development on the concepts of variable and the equal sign as well as the role these concepts play in students' abilities to solve problems.

For example, student data from the two variable tasks might be shared with teachers to illustrate misconceptions students hold about literal symbols and to demonstrate that even for students who appear to have a solid understanding of variable (e.g., by giving a multiple values interpretation of the symbol n), that knowledge may prove to be less stable when they are required to apply it (e.g., when asked to solve the *which is larger* task). Sharing research on student thinking about such items can also point to the importance of the contexts in which variables are first introduced. If, for example, students' early experiences with literal symbols occur primarily in the context of one-step equations in which they are asked to solve for *n*, they may have difficulty grasping that in other contexts *n* can take on any value—knowledge that is necessary when evaluating an expression.

Teachers should be encouraged to view items such as the *equivalent equations* task as opportunities to employ and foster an understanding of the equal sign, and not simply as an opportunity to practice standard equation-solving techniques. That only 7 of the 20 teachers mentioned *recognize equivalence* as a strategy that students might use to solve this task suggests a lack in teachers' own understandings of the mathematics involved, as opposed to simply a lack of knowledge about students' thinking. Student thinking can nevertheless serve as a vehicle for addressing such gaps in teacher knowledge. For example, teachers might be asked to study and discuss STAAR findings on the two equal sign tasks presented here after first thinking about the tasks for themselves. Such experiences, grounded in discussions of student work and student thinking, may help develop teachers' "algebra eyes and ears" (Blanton & Kaput, 2003) so that they become better able to recognize and capitalize on opportunities to develop students' algebraic thinking in the context of everyday mathematics lessons. For example, "equation strings" such as 5 + 7 = 12 + 5 = 17—often used by students and even teachers as a record-keeping device—should be recognized both as an inappropriate use of the equal sign and as an opportunity to discuss the equal sign's meaning.

Students' extensive prior school exposure to the equal sign, coupled with the lack of instructional attention the symbol receives in the teachers' curricular materials (documented by McNeil et al., 2006), were likely contributors to teachers' overestimation of students' relational understanding. In explaining their rationale for their predictions, 16 participants stated that given many years of experience with the equal sign, students must already hold a relational understanding of this symbol. Recall that Kevin, an eighth-grade teacher, predicted that "almost 99%" of students would hold a relational view "because they've used the equal sign a lot since kindergarten." The clear implication for mathematics teacher professional development is the need to distinguish between exposure and understanding. Teachers should be encouraged to consider McNeil et al.'s findings that equal signs presented in "operations on both sides" contexts (e.g., 3 + 4 = 5 + ?) —largely absent from four popular middle-school mathematics textbooks—are much more likely to elicit relational understandings than the more common "operations equals answer" contexts (e.g., 3 + 4 = ?).

Another connection teachers made to students' prior experiences was attributing students' misconceptions about multiplication to elementary school instruction. Although teachers overestimated the proportion of students who demonstrated this misconception when justifying that $3n$ is larger than $n + 6$ on the *which is larger* task, it nonetheless raises an issue about the important relationship between operation sense and algebraic thinking. A solid understanding of operations and number sense is critical to algebraic reasoning. For example, when a student has had the opportunity in elementary school "to think through what multiplication does, why 18×12 is equivalent to $18 \times 10 + 18 \times 2$ [then when that student] enters an algebra class, having had such an opportunity…he will understand why (a + b)(c + d) does not equal ac + bd" (Schifter, 1999, p. 75). Students need practice beyond memorizing rules and procedures to develop a deep level of conceptual understanding. Professional development efforts are needed that focus on connections between what has been considered the domain of arithmetic (such as learning about the equal sign and developing number sense) and the algebra learning occurring in middle school.

CONCLUSION

Teachers' responses to the four tasks presented in this article reveal a great deal about their knowledge of the development of students' understandings of two core algebraic concepts—variable and the equal sign—during the middle school years. This study highlights the importance of sharing with middle school teachers research-based knowledge about the development of students' algebraic thinking and making connections with their early math education. For example, disseminating STAAR project findings could contribute to teacher knowledge by unveiling common student misconceptions (e.g., that letters as variables are abbreviations or objects, or that the equal sign means "the answer") and lead to a discussion of how students' understanding of core algebraic concepts can contribute to the development of more sophisticated understanding (e.g., the ability to work with equations and algebraic expressions).

The role of students' prior exposure to variables and equal signs along with the explicit (in the case of variable) or lack of explicit (in the case of the equal sign) attention these concepts receive in the curriculum may have contributed to teachers' accuracy with variable task predictions and lack of accuracy with equal sign task predictions. Regardless of the source of teachers' existing knowledge, however, it would be to teachers' advantage to know more about student thinking in the domain of early algebra. Advancing teachers' knowledge of student thinking—specifically with regard to variable and the equal sign—will enable them to be more attentive to students' needs and recognize opportunities to foster understanding.

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